Important Questions of Engineering Mathematics – II

Q.1. Solve $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$

Soln. : Given differential equation is, $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$

$$(x^2+y^2+1)\,dx = xy\,dy$$

$$(x^2+y^2+1)\,dx - 2xy\,dy = o \qquad ...(1)$$

$$\therefore M = x^2+y^2+1$$

$$N = 2xy$$

$$\therefore \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{D.E. is not Exact D.E.}$$

But it is of the form,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\text{i.e. } \frac{2y-(-2y)}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\therefore \text{ I.F.} = e^{\int f(x)dx} = e^{\int \frac{-2}{x} dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$$

$$\text{I.F.} = \frac{1}{x^2}$$

Multiply Equation (1) by I.F.

$\therefore$ M and N becomes,

$$M = \frac{1}{x^2}[x^2+y^2+1] = 1 + \frac{y^2}{x^2} + \frac{1}{x^2}$$

$$N = \frac{1}{x^2}[-2xy] = -\frac{2y}{x} \qquad \therefore \frac{\partial M}{\partial y} = \frac{2y}{x^2} = \frac{\partial N}{\partial x}$$

$\therefore$ Differential equation is now exact.

Hence solution is,

$$\int_{y=\text{const}} M\,dx + \int [\text{Terms of N not containing x}]\,dy = C$$

$$\int_{y=\text{const}} \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx + \int o\,dy = C$$

$$x + y^2\left[\frac{x^{-1}}{-1}\right] + \left[\frac{x^{-1}}{-1}\right] \; o = C$$

$$\boxed{x - \frac{y^2}{x} - \frac{1}{x} = C} \quad \text{i.e.} \quad \boxed{x - \frac{y^2+1}{x} = C}$$

Q.2. Solve $\frac{dy}{dx} = \frac{x+y+3}{3x+3y-3}$

Soln. : Given differential equation is,

$$\frac{dy}{dx} = \frac{x+y+3}{3x+3y-3}$$

This is non Homogeneous differential equation,

Here, $\therefore \frac{a_1}{a_2} = \frac{1}{3}$ and $\frac{b_1}{b_2} = \frac{1}{3} \Rightarrow \therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{2}$

$$\therefore \frac{dy}{dx} = \frac{(x+y)+3}{3(x+y)-3} \quad \text{...(1)}$$

Put $x + y = u \quad \therefore 1 + \frac{dy}{dx} = \frac{du}{dx} \quad \therefore \frac{dy}{dx} = \frac{du}{dx} - 1$

$\therefore$ Differential equation (1) becomes,

$$\therefore \frac{du}{dx} - 1 = \frac{u+3}{3u-3}$$

$$\frac{du}{dx} = \frac{u+3}{3u-3} + 1 = \frac{u+3+3u-3}{3u-3} = \frac{4u}{3u-3}$$

$$\therefore \frac{3u-3}{4u} du = dx$$

This is Variable separate form,

$$\int \frac{3u-3}{4u} du = \int dx + C$$

$$\int \left(\frac{3}{4} - \frac{3}{4}u\right) du = x + C$$ After integrating -

$$\int \frac{3}{4}u - \frac{3}{4}\left(\frac{u^2}{2}\right) = x + C$$

$$\frac{3}{4}(x+y) - \frac{3}{8}(x+y)^2 = x + C$$

$$-\frac{1}{4}x + \frac{3}{4}y - \frac{3}{8}(x+y)^2 = C$$

$$2x - 6y + 3(x+y)^2 = C$$

Q.3. Solve $x\frac{dy}{dx} + y = y^2 \log x$

Soln. :

Step I : Given differential equation is,

$$x\frac{dy}{dx} + y = y^2 \log x$$

Divide throughout by x

$$\therefore \frac{dy}{dx} + \frac{1}{x}y = y^2 \frac{\log x}{x}$$

Which is Bernoullie's Linear differential Equation,

Divide throughout by y^2,

$$\therefore y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{\log x}{x}$$

Put $y^{-1} = u$

Differentiate w.r. to x,

$$\therefore \quad -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

Substitute these values in Equatrion (1),

$\therefore$ Differetial Equation (1) becomes,

$$-\frac{du}{dx}+\frac{1}{x}u = \frac{\log x}{x}$$

$$\frac{du}{dx} - \frac{1}{x}u = -\frac{\log x}{x}$$

This is Linear differential Equation of the form $\frac{du}{dx} + Pu = Q$

Step II : Here $\quad P = \frac{-1}{x} \quad$ and $\quad Q = \frac{-\log x}{x}$

$$\therefore \text{ I.F.} = e^{\int P dx}$$

$$\int P dx = \int \frac{-1}{x} dx = -\log x = \log x^{-1}$$

$$\therefore \quad \text{I.F.} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Step III : Hence the solution is,

$$u\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$$

$$u\frac{1}{x} = \int \frac{-\log x}{x}\frac{1}{x}dx + C$$

$$\frac{u}{x} = -\int \frac{\log x}{x^2}\,dx = -\int \frac{1}{x^2}\log x\,dx$$

$$= -\left\{\log x \cdot \left(\frac{x^{-1}}{-1}\right) - \int \frac{1}{x}\left(\frac{x^{-1}}{-1}\right)dx\right\} + C \qquad \text{(Integrating by parts)}$$

$$= \frac{\log x}{x} - \int \frac{1}{x^2}dx + C$$

$$\frac{u}{x} = \frac{\log x}{x} - \left(\frac{x^{-1}}{-1}\right) + C$$

$$\frac{y^{-1}}{x} = \frac{\log x}{x} + \frac{1}{x} + C$$

$$\therefore \quad \mathbf{\frac{1}{xy} = \frac{\log x + 1}{x} + C}$$ This is required solution.

Q.4. **A body of a temperature 100^0C is placed in a room whose temperature is 20^0C and cools to 60^0C in 5 minutes. Find the temperature of body after further interval of 3 minutes?**

Soln. :

Step I : We have,

By Newton's law of cooling, $\frac{d\theta}{dt} = -k(\theta - \theta_0)$.

Where, θ_0 is temperature of surrounding and θ is temperature of body at any instant.

Step II : Given, $\theta_0 = 20°C$

t min.	θ°C
0	100
5	60
8	$\theta_1 = ?$

$$\therefore \frac{d\theta}{dt} = -k(\theta - 20)$$

$$\frac{d\theta}{\theta - 20} = -k\,dt \qquad ...(1)$$

This is variable separable form,

Step III : $\int_{100}^{60} \frac{d\theta}{\theta - 20} = -k\int_0^5 dt$ (From given values)

$$[\log(\theta - 20)]_{100}^{60} = -k[t]_0^5$$

$$\log\left(\frac{40}{80}\right) = -k[5] \qquad \therefore k = -\frac{1}{5}\log\left(\frac{1}{2}\right) \qquad ...(2)$$

Step IV : Again from Equation (1),

$$\int_{100}^{\theta_1} \frac{d\theta}{\theta - 20} = -k\int_0^8 dt$$

$$[\log(\theta - 20)]_{100}^{\theta_1} = -k[t]_0^8 = -k(8)$$

$$\log\left(\frac{\theta_1 - 20}{80}\right) = \frac{1}{5}\log\left(\frac{1}{2}\right)(8) \qquad \text{[From Equation (2) value of k]}$$

$$= \frac{8}{5}\log\left(\frac{1}{2}\right) = \log\left(\frac{1}{2}\right)^{8/5}$$

$$\therefore \frac{\theta_1 - 20}{80} = \left(\frac{1}{2}\right)^{8/5}$$

$$\therefore \theta_1 = 80 \times \left(\frac{1}{2}\right)^{8/5} + 20 = 26.3902 + 20 = 46.3902$$

∴ After 8 minutes the temperature of the body will be 46.3902°C.

Q.5. **A constant e.m.f. E volts is applied to a circuit containing constant resistance R ohms in series and a constant inductance L henries. The current i at any time t is given by $L\frac{di}{dt} + Ri = E$ if the initial current is zero. Show that the current builds up to half its theoretical maximum value in $\frac{L}{R}\log 2$ seconds.**

Soln. :

Step I : Here given equation is,

$$L\frac{dI}{dt}+RI = E$$

$$\frac{dI}{dt}+\frac{R}{L}I = \frac{E}{L} \qquad \text{(Divide through out by L) ...(1)}$$

This is linear differential equation $\left(\frac{dI}{dt}+PI=Q\right)$

Step II : Here, $P=\frac{R}{L}$ and $Q=\frac{E}{L}$

$$I.F. = e^{\int P\,dt} = e^{\int R/L\,dt} = e^{Rt/L}$$

Hence the solution is,

$$I\,(I.F.) = Q\,(I.F.)\,dt + A \qquad \text{(A is integration constant)}$$

$$I\cdot e^{Rt/L} = \int\frac{E}{L}e^{Rt/L}dt + A = \frac{E}{L}\frac{e^{Rt/L}}{\frac{R}{L}} + A$$

$$I\,e^{Rt/L} = \frac{E}{R}e^{Rt/L} + A \qquad ...(2)$$

Step III : Given initial condition : When $t = 0, I = 0$

$$0 = \frac{E}{R}+A \Rightarrow A = -\frac{E}{R}$$

Substitute this value in Equation (2)

$$\therefore I\,e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R}$$

$$I = \frac{E}{R} - \frac{E}{R}\cdot e^{-Rt/L} = \frac{E}{R}[1-e^{-Rt/L}] \qquad ...(3)$$

Step IV : From Equation (3), $I_{max}=\frac{E}{R}$

$\therefore$ Half of its theoretical maximum is $\frac{1}{2}I_{max} = \frac{1}{2}\frac{E}{R}$

Put $I=\frac{1}{2}\frac{E}{R}$ in Equation (3)

$$\therefore \frac{1}{2}\frac{E}{R} = \frac{E}{R}(1-e^{-Rt/L})$$

$$\frac{1}{2} = 1-e^{-Rt/L}$$

$$\therefore e^{-Rt/L} = 1-\frac{1}{2} = \frac{1}{2}$$

$$\therefore e^{Rt/L} = 2 \qquad \text{(taking reciprocal of both sides)}$$

$$\frac{Rt}{L} = \log 2 \qquad \therefore t = \frac{L\log 2}{R}$$

Hence proved.

Q.6. A pipe 20 cm in diameter contains steam at 200^0C and is protected with a covering 5 cm thick for which k = 0.0015. If the temperature of the outer surface of the covering is 50^0C. Find the temperature half-way through the covering under steady state conditions.

Soln. :

Step I : We know, if heats loss is q cal/sec then,

$$q = -k(2\pi x)\frac{dT}{dx} \quad \ldots(1)$$

Where x is distance of surface from centre and T is temperature of surface at any distance

Step II : Given, $k = 0.0015$

x cm	T°C
10	200
15	50
12.5	T = ?

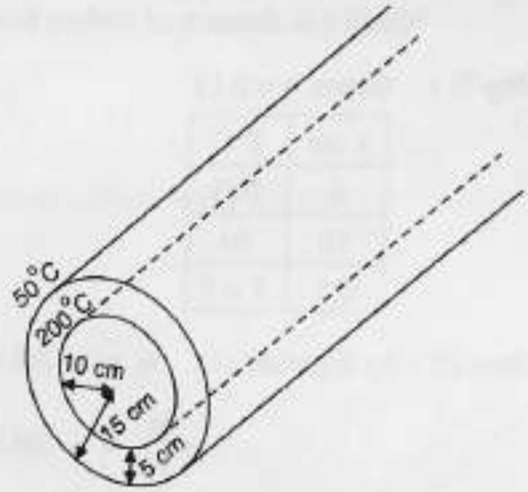


Fig. P. 2.5.3

$$q = (-2\pi k)\, x\frac{dT}{dx}$$

$$\frac{dx}{x} = (-2\pi k)\, dT$$

$$q\frac{dx}{x} = (-2\pi k)\, dT \quad \ldots(2)$$

This is variable separable form,

Step III : $q\int_{10}^{15}\frac{dx}{x} = (-2\pi k)\int_{200}^{50} dT$ (Given values from table)

$$q\,[\log x]_{10}^{15} = (-2\pi k)\,[T]_{200}^{50}$$

$$q\log\left(\frac{15}{10}\right) = -2\pi k\,[T]_{200}^{50}$$

$$q\log(1.5) = [-2\pi k]\,[50-200]$$

$$q = \frac{(-2\pi k)(-150)}{\log(1.5)} \quad \ldots(3)$$

Step IV : Again from Equation (1)

$$q\int_{10}^{12.5}\frac{dx}{x} = (-2\pi k)\int_{200}^{T_1} dT$$ (Given values from table)

$$q\,[\log x]_{10}^{12.5} = (-2\pi k)\,[T]_{200}^{T_1}$$

$$\frac{(-2\pi k)(-150)}{\log(1.5)}\left[\log\frac{12.5}{100}\right] = (-2\pi k)\,[T_1 - 200]$$ [Value of q from Equation (3)]

$$T_1 - 200 = \frac{-150\log(1.25)}{\log(1.5)}$$

$\therefore$ $\boxed{T = 117.45°C}$

Q.7. A body starts moving from rest is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the body at that instant. Show that the velocity of the body is given by, $v^2 = \frac{c}{2b^2}(1-e^{-2bx}) - \frac{cx}{b}$.

Soln. :

Step I : Here Equation of motion is,

$$\text{Force} = -\text{opposed force} - \text{resistance}$$

$$mv\frac{dv}{dx} = -mcx - mbv^2$$

$$v\frac{dv}{dx} = -cx - bv^2$$

$$v\frac{dv}{dx} + bv^2 = -cx \quad ...(1)$$

$$\text{Put } v^2 = u \Rightarrow 2v\frac{dv}{dx} = \frac{du}{dx}$$

Differential equation becomes,

$$\frac{1}{2}\frac{du}{dx} + bu = -cx$$

$$\frac{du}{dx} + 2bu = -2cx$$

This is Linear differential equation $\left(\frac{du}{dx} + Pu = Q\right)$.

Step II : Here $P = 2b$ and $Q = -2cx$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int 2b\, dx} = e^{2bx}$$

Hence the solution is,

$$u\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + A \text{(A is integration constant)}$$

$$u\, e^{2bx} = \int -2cx\, e^{2bx}\, dx + A$$

$$v^2 e^{2bx} = -2c\left\{(x)\frac{e^{2bx}}{2b} - (1)\left(\frac{e^{2bx}}{4b^2}\right)\right\} + A$$

$$v^2 e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + A \quad ...(2)$$

Step III : Given initial condition : When $t = 0$, $x = 0$, $v = 0$ (body starts moving from rest)

$$0 = 0 + \frac{c}{2b^2} + A \qquad \Rightarrow A = -\frac{c}{2b^2}$$

$$v^2 e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} - \frac{c}{2b^2}$$

[From Equation (2)]

$$v^2 = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}$$ Hence proved.

Q.8. **Find the Fourier series to represents the function** $f(x) = \pi^2 - x^2$ **in the interval** $-\pi \le x \le \pi$ **and** $f(x+2\pi) = f(x)$

Deduce that :

(i) $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ----- = \frac{\pi^2}{12}$$

(ii) $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ----- = \frac{\pi^2}{8}$$

Soln. : Since interval is of the form $-a < x < a$; we check whether the function f(x) is even or odd.

Since
$$f(x) = \pi^2 - x^2$$
$$f(-x) = \pi^2 - (-x)^2$$
$$f(-x) = \pi^2 - x^2$$

$\therefore$ $f(-x) = f(x)$, given function is even function. Therefore fourier series of f(x) given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

here $l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Step I :
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left\{\left(\pi^3 - \frac{\pi^3}{3}\right) - (0-0)\right\} = \frac{2}{\pi}\left[\frac{2\pi^3}{3}\right] \quad \text{...(1)}$$
$$a_0 = \frac{4\pi^2}{3}$$

Step II :
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx$$
$$= \frac{2}{\pi}\left[(\pi^2 - x^2)\left(\frac{\sin nx}{n}\right) - (-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right) - 0 \ldots\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\left\{\frac{1}{n}(\pi^2 - x^2)\sin nx - \frac{2}{n^2}x\cos nx + \frac{2}{n^3}\sin nx\right\}\right]_0^{\pi}$$

since $(\sin nx)_0^{\pi} = 0$

$$\therefore a_n = \frac{2}{\pi}\left\{0 - \frac{2}{n^2}(x\cos nx)_0^{\pi} + 0\right\} = \frac{2}{\pi}\left(-\frac{2}{n^2}\right)(\pi\cos n\pi - 0) = -\frac{4}{\pi n^2}\pi(-1)^n = -\frac{4}{n^2}(-1)^n$$

Step III : $\therefore$ Using values a_0 and a_n in Equation (1) we get,

$$(\pi^2 - x^2) = \frac{1}{2}\left(\frac{4\pi^2}{3}\right) + \sum_{n=1}^{\infty} -\frac{4}{n^2}(-1)^n\cos nx$$

$$\pi^2 - x^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \quad \text{...(2)}$$

This is Fourier series of f(x)

Step IV : Put $x = 0$ in Equation (2) we get

$$\pi^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$
$$\pi^2 - \frac{2\pi^2}{3} = -4\left[-\frac{1}{1^2} + \frac{1}{2^2} + \left(\frac{-1}{3^2}\right) + \frac{1}{4^2} + \ldots\right]$$
$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \ldots\right]$$
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12} \quad \text{...(3)}$$

Step V : Put $x = \pi$ in Equation (2)

$$\pi^2 - \pi^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$
$$0 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$
$$-\frac{2\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{1}{n^2} \quad (\because (-1)^{2n} = 1)$$
$$\left(-\frac{2\pi^2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \ldots\ldots$$
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \ldots\ldots$$
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \ldots = \frac{\pi^2}{6} \quad \text{...(4)}$$

Step VI : Adding Equation (3) and Equation (4) we get

$$\frac{2}{1^2} + 0 + \frac{2}{3^2} - 0 + \frac{2}{5^2} - \ldots = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$
$$\Rightarrow 2\left[\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{12^2}\right] = \frac{\pi^2}{4}$$
$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \frac{\pi^2}{8} \quad \text{...(5)}$$

Q.9. **Find the Fourier series of following values up to second harmonics in the interval (0,6)**

x	0	1	2	3	4	5	6
y	4	8	15	7	6	2	4

Soln. : Here length of interval $2l = 6$

$\therefore$ Fourier series of f(x) in the interval (0, 6) with period T = 6 is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}$$

where $a_0 = 2 \times$ mean of values of y

$a_n = 2 \times$ mean of values of $y \cos\frac{n\pi x}{3}$

$b_n = 2 \times$ mean of values of $y \sin\frac{n\pi x}{3}$

x	y	$y\cos\left(\frac{\pi x}{3}\right)$	$y\sin\left(\frac{\pi x}{3}\right)$	$y\cos\left(\frac{2\pi x}{3}\right)$	$y\sin\left(\frac{2\pi x}{3}\right)$
0	4	4.0	0.0	4.0	0.0
1	8	4.0	6.9	– 4.0	6.9
2	15	– 7.5	13.0	– 7.5	– 13.0
3	7	– 7	0.0	7.0	0.0
4	6	– 3	– 5.2	– 3.0	5.2
5	2	1	– 1.7	– 1.0	– 1.7
Σ	42	– 8.5	13.0	– 4.5	– 2.6

$$\therefore a_0 = 2 \times \frac{\Sigma y}{6} = 2 \times \frac{42}{6} = 14$$

$$a_1 = 2 \times \text{mean of values of } y \cos\frac{\pi x}{3} = 2 \times \frac{\Sigma y \cos\frac{\pi x}{3}}{6} = 2 \times \left(\frac{-8.5}{6}\right) = -2.833$$

$$b_1 = 2 \times \text{mean of values of } y \sin\frac{\pi x}{3} = 2 \times \frac{\Sigma y \sin\frac{\pi x}{3}}{6} = 2 \times \left(\frac{13.0}{6}\right) = 4.34$$

$$a_2 = 2 \times \text{mean of values of } y \cos\frac{2\pi x}{3} = 2 \times \left(\frac{-4.5}{6}\right) = -1.5$$

$$b_2 = 2 \times \text{mean of values of } y \sin\frac{2\pi x}{3} = 2 \times \left(\frac{-2.6}{6}\right) = -0.867$$

$\therefore$ Required Fourier series of y is given by,

$$y = \frac{1}{2}(14) + (-2.833)\cos\frac{\pi x}{3} + (4.34)\sin\frac{\pi x}{3} + (-1.5)\cos\frac{2\pi x}{3} + (-0.867)\sin\frac{2\pi x}{3}$$

$$= 7 + \left(-2.833\cos\frac{\pi x}{3} + 4.34\sin\frac{\pi x}{3}\right) + \left(-1.5\cos\frac{2\pi x}{3} - 0.867\sin\frac{2\pi x}{3}\right) \ldots\ldots$$

Q.10. **If** $I_n = \int_0^{\frac{\pi}{4}} sec^n\,\theta\,d\theta$ **, prove that** $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$. **Hence evaluate** $\int_0^{\frac{\pi}{4}} sec^6\,\theta\,d\theta$

Soln. :

$$I_n = \int_0^{\pi/4} \sec^n\theta\,d\theta = \int_0^{\pi/4} \sec^{n-2}\theta\sec^2\theta\,d\theta$$

$$= \left[\sec^{n-2}\theta\tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4}(n-2)\sec^{n-3}\theta\,(\sec\theta\tan\theta)\tan\theta\,d\theta$$

$$= \left(\sqrt{2}\right)^{n-2} - (n-2)\int_0^{\pi/4}\sec^{n-2}\theta\tan^2\theta\,d\theta \qquad \left(\because \sec\frac{\pi}{4} = \sqrt{2}\right)$$

$$= \left(\sqrt{2}\right)^{n-2} - (n-2)\left[\int_0^{\pi/4}\sec^{n-2}\theta\,(\sec^2\theta - 1)\,d\theta\right]$$

$$I_n = \left(\sqrt{2}\right)^{n-2} - (n-2)\,I_n + (n-2)\,I_{n-2}$$

$$(1+n-2)\,I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)\,I_{n-2}$$

$$\Rightarrow (n-1)\,I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)\,I_{n-2}$$

$$I_n = \frac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \frac{n-2}{n-1}\,I_{n-2} \qquad \text{...(1)}$$

Hence proof

Now, put n = 6 in Equation (1)

$$I_6 = \frac{\left(\sqrt{2}\right)^{4}}{5} + \frac{4}{5}\,I_4 = \frac{4}{5} + \frac{4}{5}\,I_4 \qquad \text{...(2)}$$

Put n = 4 in Equation (1) ,

$$I_4 = \frac{\left(\sqrt{2}\right)^{2}}{3} + \frac{2}{3}\,I_2 = \frac{2}{3} + \frac{2}{3}\,I_2 \qquad \text{...(3)}$$

Put n = 2 in Equation (1)

$$I_2 = \frac{1}{1} + 0 = 1 \qquad \text{...(4)}$$

$$\therefore\ I_4 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \text{ (From Equation (3))}$$

$$I_6 = \frac{4}{5} + \frac{4}{5}\times\frac{4}{3} = \frac{4}{5} + \frac{16}{15} = \frac{12+16}{15} = \mathbf{\frac{28}{15}}$$

Q.11. Evaluate $\int_3^5 (x-3)^{\frac{1}{2}} (5-x)^{\frac{1}{2}} dx$

Soln. : Let $I = \int_3^5 (x-3)^{\frac{1}{2}}(5-x)^{\frac{1}{2}} dx$ ...(1)

Put $x - 3 = (5-3)t$

i.e. $x - 3 = 2t$

$\therefore x = 3 + 2t \Rightarrow dx = 2dt$

x	3	5
$T = \frac{x-3}{2}$	0	1

$\therefore$ From Equation (1)

$$I = \int_0^1 (2t)^{\frac{1}{2}} [5-(3+2t)]^{\frac{1}{2}}\ 2\,dt = \int_0^1 (2t)^{\frac{1}{2}} [2-2t]^{\frac{1}{2}}\ 2\,dt$$

$$= \int_0^1 2^{\frac{1}{2}}\, t^{\frac{1}{2}}\, [2(1-t)]^{\frac{1}{2}}\ 2\,dt = 2^{\frac{1}{2}} (2) \int_0^1 t^{\frac{1}{2}}\, 2^{\frac{1}{2}} (1-t)^{\frac{1}{2}}\ dt$$

$$= 2^{\frac{1}{2}} (2)\, 2^{\frac{1}{2}} \int_0^1 t^{\frac{1}{2}} (1-t)\, dt$$

$$= 4\beta\left(\frac{1}{2}+1, \frac{1}{2}+1\right) \qquad \left(\because \int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1)\right)$$

$$= 4\beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= 4\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} \qquad \left(\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right)$$

$$= 4\frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)\left(\frac{1}{2}\Gamma\frac{1}{2}\right)}{\Gamma 3} \qquad \left(\because \Gamma n = (n-1)\Gamma(n-1)\right) \text{ if n is fraction}$$

$$= 4\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}\sqrt{\pi}}{2!} \qquad \left(\Gamma\frac{1}{2} = \sqrt{\pi} \text{ and } \Gamma n = (n-1)! \text{ if n is positive integer}\right)$$

$$= \frac{\pi}{2}$$

Q.12. If $\phi(a) = \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$; then find $\phi'(a)$ and show that $\phi(a)$ is independent of a.

Soln. :

Step I : Given, $\phi(a) = \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$ ...(1

Here, a - is parameter; x-is the variable of integration.

Upper and lower limits are functions of parameter a.

Step II : By DUIS, rule – II, differentiating Equation (2) w.r.t. parameter a, keeping x as constant.

$$\frac{d\phi(a)}{da} = \frac{d}{da}\int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$$

$$= \int_{\pi/6a}^{\pi/2a} \left[\frac{\partial}{\partial a}\cdot\frac{\sin ax}{x}\right]\cdot dx + \left[\frac{\sin ax}{x}\right]_{x=\frac{\pi}{2a}} \cdot \frac{d}{da}\left(\frac{\pi}{2a}\right) - \left[\frac{\sin ax}{x}\right]_{x=\frac{\pi}{6a}} \cdot \frac{d}{da}\left(\frac{\pi}{6a}\right)$$

$$\frac{d\phi(a)}{da} = \int_{\pi/6a}^{\pi/2a} \frac{x\cdot\cos ax}{x} dx + \frac{\sin(\pi/2)}{(\pi/2a)}\cdot\left(\frac{-\pi}{2a^2}\right) - \frac{\sin \pi/6}{(\pi/6a)}\left(-\frac{\pi}{6a^2}\right)$$

Integrating first term w.r.t. x

Step III: $= \left(\frac{\sin ax}{a}\right)_{\pi/6a}^{\pi/2a} - \frac{1}{a}\cdot(1) + \frac{1}{a}\cdot\left(\frac{1}{2}\right) = \frac{\sin(\pi/2)}{a} - \frac{\sin(\pi/6)}{a} - \frac{1}{a} + \frac{1}{2a}$

$$\frac{d\phi(a)}{da} = \frac{1}{a} - \frac{1}{2a} - \frac{1}{a} + \frac{1}{2a} = 0$$

$\Rightarrow$ $\frac{d\phi}{da} = 0$, means $\phi(a)$ is independent of a

Q.13. Trace the curve, $ay^2 = x^2(a - x)$

(I) **Symmetry :** This curve is symmetric about X-axis. (Since, ay have even powers)

(II) **Points of intersection :**

(i) **With X-axis :** Put $y = 0$, we get $0 = x^2(a - x) \Rightarrow x = 0, x = a$.
Points of intersection with X-axis is (0, 0) (a, 0). So there is a loop between (0, 0) to (a, 0).

(ii) **With Y-axis :** Put $x = 0$, we get, $ay^2 = 0 \Rightarrow y = 0$
$\therefore$ This the given curve intersects Y-axis is (0, 0)

(iii) **Origin :** Put $x = 0, y = 0$, we get $0 = 0$. This curve passes through origin.

(iii) **Equation of tangent :**

(i) **At origin :** By equating the lowest degree term or terms to zero.

$$a(y^2 - x^2) = 0$$
$$\Rightarrow y = \pm x$$

$y = \pm x$ are a tangents at origin

(ii) **At (a, 0) :**

$$\therefore ay^2 = x^2(a - x)$$
$$a\cdot 2y\frac{dy}{dx} = 2ax - 3x^2$$
$$\frac{dy}{dx} = \frac{2ax - 3x^2}{2ay}$$
$$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$$

Thus at (a, 0) curve have tangent parallel to Y-axis.

(IV) Asymptote :

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero.

$-1 = 0$ (meaningless)

No asymptote parallel to X-axis.

(ii) **Parallel to Y-axis :** Equating the co-efficient of highest degree term in y to zero.

i.e. $a = 0$

No asymptote parallel to Y-axis.

(V) Region of absence :

Since, the given curve is symmetric to X-axis so solve for Y.

$$y^2 = \frac{x^2(a-x)}{a}$$

at $x = 0$; $y^2 = 0$

$x = a$; $y^2 = 0$

$x > a$; y^2 – is negative

and $x < 0$; y^2 is always positive

Hence, curve exist in $x < 0$ and $0 \le x \le a$.

The rough sketch of the given curve is Fig. P.7.4.7.

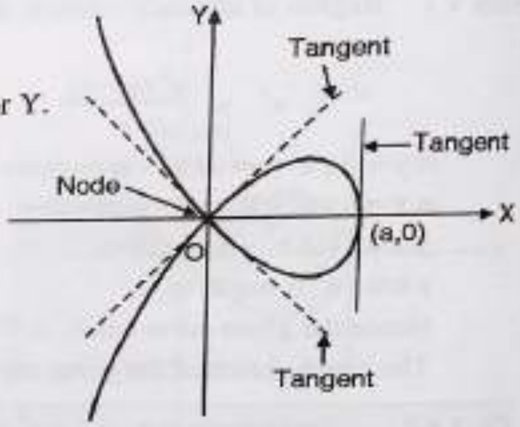


Fig. P. 7.4.7

Q.14. Trace the curve $r = a(1 + \cos\theta)$

Step I : **Symmetry :** This curve is symmetric about initial line $\theta = 0$

Step II : $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

Since $r = a(1 + \cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \quad \tan\phi = \frac{a(1+\cos q)}{-a\sin q} = \frac{(1+\cos q)}{-\sin q}$$

Step III : Table :

Form a table for r, θ, tan φ

θ	0	π/4	π/2	3π/4	π
$r = a(1+\cos\theta)$	2a	$a\left(1+\frac{1}{\sqrt{2}}\right)$	a	$a\left(1-\frac{1}{\sqrt{2}}\right)$	0
$\tan\phi = \frac{1+\cos\theta}{-\sin\theta}$	∞	$\frac{1+1/\sqrt{2}}{-1/\sqrt{2}}$	$\frac{-1}{2}$	$\frac{1-1/\sqrt{2}}{-1/\sqrt{2}}$	0

Step IV : Pole / tangent at pole :

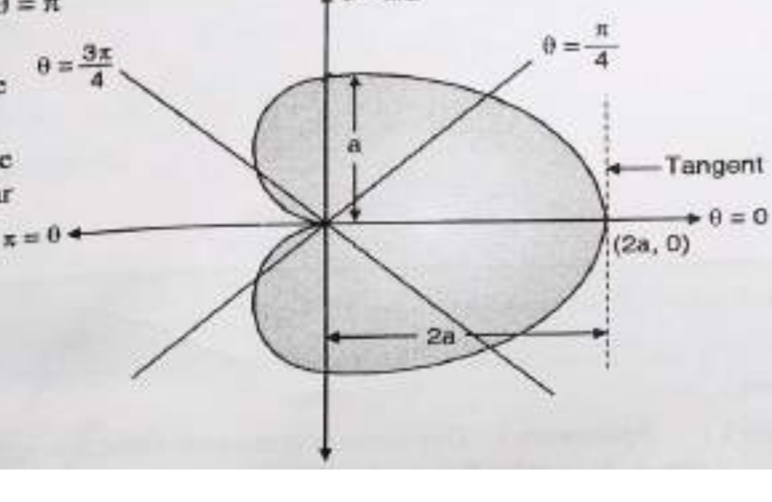
Here, $r = 0$ for $\theta = \pi$. Hence, curve passes through pole at $\theta = \pi$.

The curve have tangent at pole at $\theta = \pi$

Also, $\tan\phi = 0$ as $\theta = \pi$

So the curve have tangent coincide with radius vector at $\theta = \pi$

And $\tan\phi = \infty$ at $\theta = 0$, so the curve have tangent perpendicular to radius vector at $\theta = 0$.



Q.15. **Find the length of an arc of a cardiode : r = a (1 – cos θ) which lies outside the circle r = a cosθ.**

Step I : Let s-be the arc length of the curve $r = a(1-\cos\theta)$

$$\therefore s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad ...(1)$$

Step II : To find the length of arc of cardiode which lies outside the circle.

$$\therefore r = a\cos\theta \Rightarrow r = a\cdot\frac{x}{r}$$

$$...(\because x = r\cos\theta,\ y = r\sin\theta)$$

$$\Rightarrow r^2 - ax = 0 \qquad ...(\because x^2+y^2=r^2)$$

$$\Rightarrow x^2+y^2-ax = 0$$

$$\left(x-\frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2$$

is a circle having centre at $\left(\frac{a}{2}, 0\right)$

These two curves intersects at $\theta = \pi/3$.

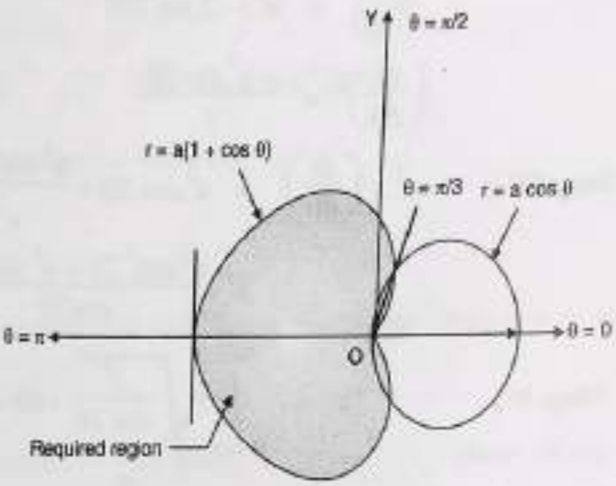


Fig. P. 7.13.8

Step III : For point of intersection,

$$\text{Since,}\quad r = a(1-\cos\theta);\ r = a\cos\theta$$

$$\therefore a\cos\theta = a(1-\cos\theta)$$

$$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3$$

Step IV : Since, $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta = a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta$$

$$= 2a^2(1-\cos\theta) = 2a^2\cdot 2\sin^2\frac{\theta}{2} = 4a^2\sin^2\frac{\theta}{2}$$

Step V :

$$\therefore s = 2\int_{\theta_1=\pi/3}^{\pi} \sqrt{4a^2\sin^2\frac{\theta}{2}}\, d\theta \qquad (\because \text{curve is symmetrical to initial line})$$

$$= 2\cdot 2a\int_{\pi/3}^{\pi} \sin\frac{\theta}{2}\, d\theta = 4a\left(\frac{-\cos\theta/2}{1/2}\right)_{\pi/3}^{\pi} = -8a\left(0-\frac{\sqrt{3}}{2}\right)$$

$$s = 4\sqrt{3}\cdot a$$

Q.16. **Find the equation of the sphere which passes through the point (3, 1, 2) and meets XOY plane in circle of radius 3 units having centre as (1, -2, 0).**

Here we find co-ordinates of centre of sphere then by using centre – radius form we find equation of sphere.

Given circle is section of required sphere taken by the XY plane

Centre of circle is $P \equiv (1, -2, 0)$ radius of circle $l(PQ) = 3$

Line CP is normal to XY plane i.e. To plane $Z = 0$

$\therefore$ dr's of line CP are 0, 0, 1.

P is centre of circle with co-ordinates $(1, -2, 0)$

$\therefore$ Equation of line CP passing through point $(1, -2, 0)$ and dr's 0,0,1 is

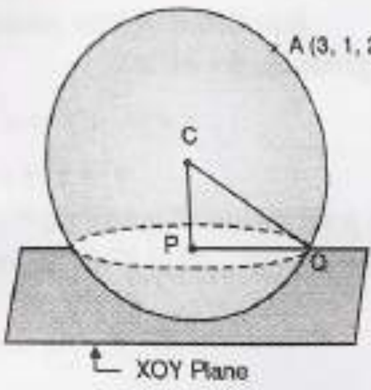


Fig. P. 8.9.4

$$\frac{x-1}{0} = \frac{y-(-2)}{0} = \frac{z-0}{1} = t \text{ say}$$

$\therefore x = 1, \quad y = -2, \quad z = t$

These are co-ordinates of any general point on line CP.

Let these are co-ordinates of point C

$\therefore C \equiv (1, -2, t)$ ... (1)

From right angled triangle Δ CPQ

$l(CP)^2 + l(PQ)^2 = l(CQ)^2$ ... (2)

Now $l(CP)^2 = (1-1)^2 + (-2+2)^2 + (t-0)^2$ ($\because$ By distance formula)

$l(CP)^2 = t^2$ ... (3)

$l(PQ) =$ Radius of circle

$= 3$ (Given) ... (4)

$\therefore$ From Equation (2)

$t^2 + 9 = l(CQ)^2$ ... (5)

$l(CA) = l(CQ)$ ($\because$ Both are equal to Radius of sphere)

$\therefore l(CA)^2 = l(CQ)^2$

$(1-3)^2 + (-2-1)^2 + (t-2)^2 = t^2 + 9$

$4 + 9 + t^2 - 4t + 4 = t^2 + 9$

$\therefore 4t = 8 \qquad \therefore t = 2$

$\therefore$ Co-ordinates of point C are given by

$C \equiv (1, -2, 2)$ ($\therefore$ From Equation (1))

From equation (5)

$l(CQ)^2 = 4 + 9 = 13$

$\therefore$ Radius of sphere $= l(CQ) = \sqrt{13}$

$\therefore$ Equation of sphere by using centre-radius form is given by

$(x-1)^2 + (y+2)^2 + (z-2)^2 = 13$

i.e. $x^2 + y^2 + z^2 - 2x + 4y - 4z - 4 = 0$

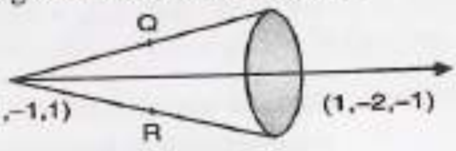
Q.17. Find the equation of a right circular cone whose vertex is (1, -1, 1), the axis parallel to $x = -\frac{y}{2} = -z$ and one of its generator has directions cosines proportional to 2, 2, 1.

Soln. : Since axis of cone is parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{-1}$, therefore dr's of axis are 1, – 2, – 1.

dr's of one of generator are proportional to 2,2,1 i.e. dr's of generator are 2,2,1.

Let α be semi vertical angle i.e. angle between one of the generator and axis of cone.

$$\therefore \cos\alpha = \frac{(1)(2) + (-2)(2) + (-1)(1)}{\sqrt{1+4+1}\sqrt{4+4+1}} = -\frac{3}{\sqrt{6}.(3)}$$



$$\cos\alpha = -\frac{1}{\sqrt{6}} \text{ i.e. } \alpha = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right)$$

Fig. P. 9.4.8

Now we have,

(i) vertex of cone at (1, – 1, 1) (ii) dr's of axis are 1, – 2, – 1

(iii) semi vertical angle is $\alpha = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right)$

Let Q (x, y, z) be any point on cone $\therefore$ dr's of generator i.e. dr's of PQ are $x-1, y+1, z-1$

$$\therefore \cos\alpha = \frac{1(x-1) - 2(y+1) - 1(z-1)}{\sqrt{1+4+1}\sqrt{(x-1)^2 + (y+1)^2 + (z-1)^2}}$$

$$-\frac{1}{\sqrt{6}} = \frac{x - 2y - z - 2}{\sqrt{6}\sqrt{(x-1)^2 + (y+1)^2 (z-1)^2}}$$

Squaring and taking cross product we get

$$(x-1)^2 + (y+1)^2 + (z-1)^2 = (x - 2y - z - 2)^2$$

$$\Rightarrow (x^2 - 2x + 1) + (y^2 + 2y + 1) + (z^2 - 2z + 1) = x^2 + 4y^2 + z^2 + 4 - 4xy - 2xz - 4x + 4yz + 8y + 4z$$

$$\Rightarrow 3y^2 - 4xy + 4yz - 2xz - 2x + 6y + 6z + 1 = 0$$

Q.18. **Find the equation of the right circular cylinder of radius 5 and axis is** $\frac{x-2}{3} = \frac{y-3}{1} = \frac{z+1}{1}$

Soln. : Equation of axis of right circular cylinder is $\frac{x-2}{3} = \frac{y-3}{1} = \frac{z+1}{1}$

$\therefore$ Axis pass through the point (2, 3, –1) and direction ratios of axis are 3, 1, 1

Here (i) Direction ratios of axis are 3, 1, 1

(ii) Axis passes through the point A(2, 3, –1)

(iii) Radius of cylinder is 5

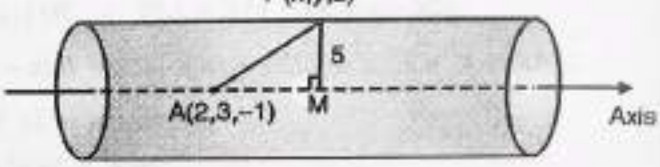


Fig. P. 9.6.4

Let P (x, y, z) be any point on the cylinder

From Fig. P. 9.6.4,

$$l(AP) = \sqrt{(x-2)^2 + (y-3)^2 + (z+1)^2}$$

$$l(PM) = 5 \text{ (Radius of cylinder)}$$

$l(AM)$ is projection of line segment AP on axis

Direction ratios of axis are 3, 1, 1

$\therefore$ Direction cosines of axis are

$$\frac{3}{\sqrt{9+1+1}}, \frac{1}{\sqrt{9+1+1}}, \frac{1}{\sqrt{9+1+1}}$$

$$\text{i.e. } \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$$

$$\therefore l(AM) = \frac{3}{\sqrt{11}}(x-2) + \frac{1}{\sqrt{11}}(y-3) + \frac{1}{\sqrt{11}}(z+1) = \frac{3x + y + z - 8}{\sqrt{11}}$$

From the right angled triangle Δ PMA

$$l(AM)^2 + l(PM)^2 = l(AP)^2$$

$$\left(\frac{3x + y + z - 8}{\sqrt{11}}\right)^2 + (5)^2 = (x-2)^2 + (y-3)^2 + (z+1)^2$$

$$\Rightarrow (3x + y + z - 8)^2 + (11)(25) = 11[(x-2)^2 + (y-3)^2 + (z+1)^2]$$

$$[9x^2 + y^2 + z^2 + 64 + 6xy + 6xz - 48x + 2yz - 16y - 16z] + 275$$

$$= 11[(x^2 - 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1)]$$

$$\Rightarrow 9x^2 + y^2 + z^2 + 6xy + 6xz + 2yz - 48x - 16y - 16z + 339$$

$$= 11x^2 + 11y^2 + 11z^2 - 44x - 66y + 22z + 154$$

$$\Rightarrow 2x^2 + 10y^3 + 10z^2 - 6xy - 6xz - 2yz + 4x - 50y + 38z - 185 = 0$$

Q.19. **Evaluate** $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dx\,dy$ **by changing the order of integration.**

Soln. : Step I : Let, $I = \int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dx\,dy$

Step II : Here the limits inner integrals are the functions of x, so these limits are of y

$\therefore$ The given limits are :

$$y = x \;;\; y = \infty$$

$$x = 0 \;;\; x = \infty$$

The required bounded region is as shown in Fig. P.10.6.6 (Shaded part)

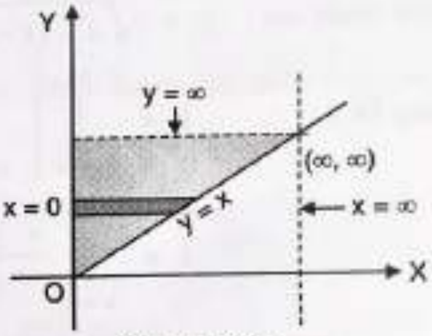


Fig. P.10.6.6

Step III : After changing the order of integration now, the limits of inner integral must interms of y,

$\therefore$ By using horizontal strip (parallel to X-axis)

New limits are : $x_1 = 0$ to $x_2 = y$

and $y_1 = 0$ to $y_1 = \infty$

Step IV : $\therefore \quad I = \int_{y=0}^{\infty}\left[\int_{x_1=0}^{y}\frac{e^{-y}}{y}\,dx\right]dy$

Step V : $I = \int_{y=0}^{\infty}\frac{e^{-y}}{y}(x)_{x=0}^{y}\,dy = \int_{y=0}^{\infty}\frac{e^{-y}}{y}(y)\,dy = \int_{y=0}^{\infty}e^{-y}\,dy = \left(\frac{e^{-y}}{-1}\right)_0^{\infty} = -[0-1] = 1$

Q.20. **Evaluate** $\iiint_v\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)}\,dx\,dy\,dz$**, where, v is the volume of the ellipsoid:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Soln. : Step I : Let, $I = \iiint\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)}\,dx\,dy\,dz$

Step II : By elliptical co-ordinates

$$x = ar\sin\theta\cos\phi \;;\; y = br\sin\theta\sin\phi \;;\; z = cr\cos\theta$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = r^2$$

$$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Then ellipsoid gets transformed to $x^2+y^2+z^2 = 1$ (unit sphere)

Limits are, $r = 0$ to 1 ; $\theta = 0$ to π and $\phi = 0$ to 2π

Step III : $\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1}\sqrt{1-r^2}\cdot abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$

$$= abc\int_{\phi=0}^{2\pi}d\phi\int_{\theta=0}^{\pi}\sin\theta\,d\theta\cdot\int_{r=0}^{1}r^2\sqrt{1-r^2}\,dr$$

Step IV : For inner most integral ; Put $r = \sin t$, $dr = \cos t$

as $r = 0 \;;\; t = 0$

$r = 1 \;;\; t = \pi/2$

Step V : $\therefore \quad I = abc\int_{\phi=0}^{2\pi}d\phi\cdot\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{t=0}^{\pi/2}\sin^2 t\cdot\cos^2 t\,dt$

$$= abc\,(\phi)_0^{2\pi}\,(-\cos\theta)_0^{\pi}\left(\frac{1.1}{4.2}\cdot\frac{\pi}{2}\right) = \frac{\pi}{16}\,abc\,(2\pi)\,(1+1)$$

$$\boxed{I = \frac{\pi\,abc}{4}}$$

Q.21. Find the total area bounded between two cardioids: $r = a(1+\cos\theta)$ **and** $r = a(1-\cos\theta)$.

Step I :

Since, Area(A) $= \iint r\,dr\,d\theta$

Step II :

The given curve are,

$$r = a(1+\cos\theta)\ ; \text{ and}$$
$$r = a(1-\cos\theta),$$

Step III : By the knowledge of curve tracing, we get,

The required area which is symmetrical to both axis. So, complete area is equal to 4 times, the area lies in first quadrant.

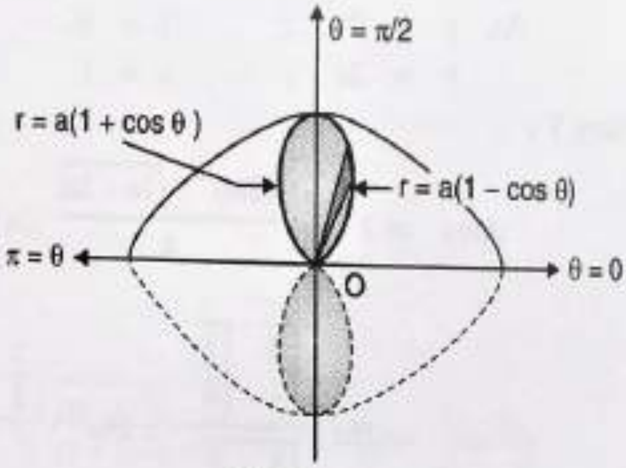


Fig. P. 11.2.10

Step IV $\therefore$ Area $= 4 \iint r\,dr\,d\theta$

$$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta = 4 \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= 2 \int_{\theta=0}^{\frac{\pi}{2}} a^2(1-\cos\theta)^2\,d\theta = 2a^2 \int_{\theta=0}^{\frac{\pi}{2}} (1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$= 2a^2\left[(\theta - 2\sin\theta)_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}\right] = 2a^2\left[\left(\frac{\pi}{2} - 2.1\right) + \frac{\pi}{4}\right]$$

$$\boxed{\text{Area} = 2a^2\left(\frac{3\pi}{4} - 2\right)}$$

Q.22. Find the volume of the cylinder $x^2 + y^2 = 2ax$ intersected between the paraboloid $x^2 + y^2 = 2az$ and the XY – plane.

Soln. :

Step I : Since, Volume (V) $= \iiint dx\,dy\,dz$

Step II : The required volume is bounded by,

$x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2az$

i.e. $(x-a)^2 + (y-0)^2 = a^2$ and $x^2 + y^2 = 2az$

or

Step III : by Cylindrical Co-ordinates

$x = r\cos\theta;\quad y = r\sin\theta;\ z = z$

$x^2 + y^2 = r^2;\quad dx\,dy\,dz = r\,dr\,d\theta\,dz$

given curves are :

$$r = 2a\cos\theta;\quad \frac{r^2}{2a} = z$$

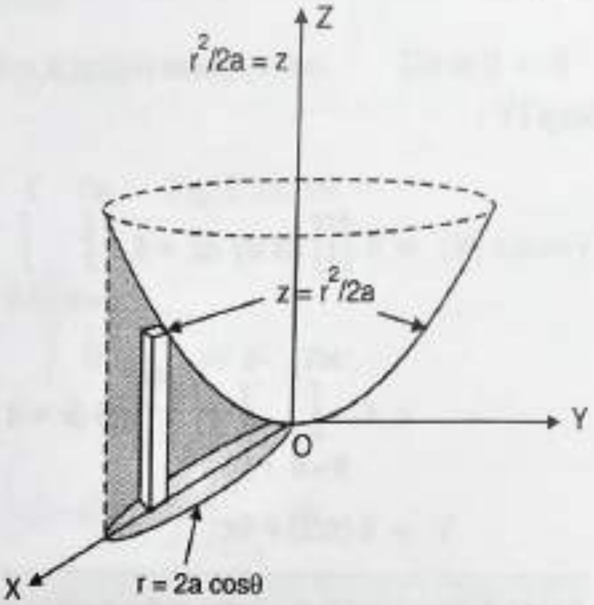


Fig. P. 11.3.9(a)

(On x-y plane, the bounded region is symmetric about X-axis)

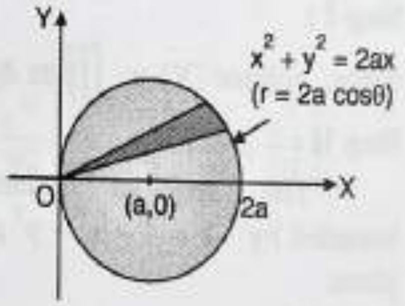


Fig. P. 11.3.9(b)

Step IV :

$$\therefore \text{Volume (V)} = 2x \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \int_{z=0}^{z=\frac{r^2}{2a}} r\,dz\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} [z]_{z=0}^{\frac{r^2}{2a}} \cdot dr \cdot d\theta$$

Step V :

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r\left(\frac{r^2}{2a}\right) dr\,d\theta = \frac{1}{a}\int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4}\right)_{r=0}^{2a\cos\theta} d\theta$$

$$= \frac{1}{4a}\int_{\theta=0}^{\pi/2} (2a)^4 \cdot \cos^4\theta\,d\theta = \frac{16a^4}{4a}\left(\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right)$$

$$\boxed{V = \frac{3\pi a^3}{4}}$$

Q.23. Find the x coordinate of center of gravity of an area bounded by the parabola $y^2 = x$ and the line $x + y = 2$

Step I : Let, $(\bar{x}, \bar{y})$ – be the C.G. of the area bounded by $y^2 = x$ and $x + y = 2$

Step II :

The point of intersection : $y^2 = x$ and $x + y = 2$ is

$$y^2 + y - 2 = 0$$

$$\Rightarrow \quad y^2 + 2y - y - 2 = 0$$

$$\Rightarrow \quad y = -2, 1$$

$$y = -2; \quad x = 4$$

$$y = 1; \quad x = 1$$

The point of intersection A (1, 1) and B (4, – 2)

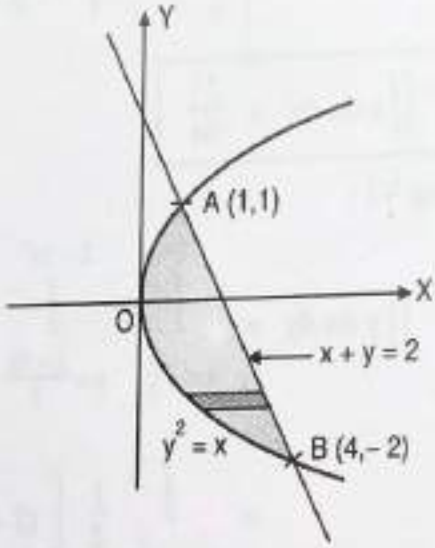


Fig. P. 11.5.14

Step III :

$$\therefore \quad \bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}; \qquad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

Here, ρ = constant = 1

To final only $\bar{x}$.

Step IV :

$$\therefore \quad \iint x\,dx\,dy = \int_{y=-2}^{1} \int_{x=y^2}^{2-y} x\,dy\,dx = \int_{y=-2}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{2-y} dy$$

$$= \int_{y=-2}^{1} [(2-y)^2 - y^4]\,dy = \frac{1}{2}\left[\frac{(2-y)^3}{-3} - \frac{y^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(-\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{4^3}{3} + \frac{2^5}{5}\right)\right] = \frac{1}{2}\left[\frac{216}{15}\right] = \frac{36}{5}$$

Step V :

$$\therefore \text{ and, } \iint dx\,dy = \int_{y=-2}^{1} \int_{x=y^2}^{2-y} y\,dy\,dx = \int_{y=-2}^{1} [2 - y - y^2]\,dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$\boxed{\iint dx\,dy = \frac{9}{2}}$$

Step VI : $\quad \therefore \bar{x} = \dfrac{\iint x\,dx\,dy}{\iint dx\,dy} = \dfrac{\left(\frac{36}{5}\right)}{\left(\frac{9}{2}\right)} = \dfrac{8}{5}$

Step VII : $\therefore$ x-co-ordinate of C.G. is $= \dfrac{8}{5}$

Q.24. **Prove that the moment of inertia of the area included between the curves** $y^2 = 4ax$ and $x^2 = 4ay$ **about x-axis is** $\frac{144}{35} M a^2$ **, where** M **is the mass of the area included between the curves.**

Step I : The given curves are $y^2 = 4\,ax$ and $x^2 = 4\,ay$

The common area between these two curve is as shown below,

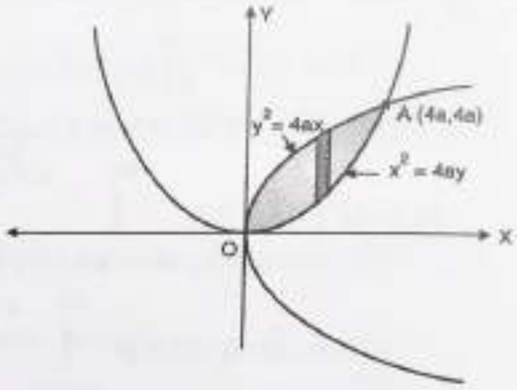


Fig. P.11.8.2

Step II : The point of intersection

$y^2 = 4\,ax$ and $x^2 = 4\,ay$

$$\Rightarrow \quad y^2 = 16\,a^2 x^2$$

$$= 16\,a^2 \cdot (4\,ay)$$

$$\Rightarrow \quad y\,(y^3 - 64\,a^3) = 0$$

$$y = 0 \text{ and } y = 4a$$

$$\Rightarrow x = 0 \text{ and } x = 4a$$

The point of intersection (0, 0) to (4a, 4a)

Here $p^2 = y^2$ ($\because$ About X-axis)

Step III :

$$\text{M.I. or } I = \iint \rho\, y^2\, dx\, dy$$

Limits are : $y = \dfrac{x^2}{4a}$ to $y = 2\sqrt{ax}$

$x = 0$ to $x = 4a$

$$\therefore \quad I = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} y^2\, dy\, dx = \rho \int_{x=0}^{4a} \left[\frac{y^3}{3}\right]_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \frac{\rho}{3} \int_{\theta=0}^{4a} \left(8\ a^{3/2} \cdot x^{3/2} - \frac{x^6}{64\,a^3}\right) dx = \frac{\rho}{3}\left[8\ a^{3/2} \cdot \frac{x^{5/2}}{5/2} - \frac{x^7}{7 \times 64\,a^3}\right]_{x=0}^{4a}$$

$$= \rho/3 \left[\frac{16}{5} a^{3/2} \cdot (4a)^{5/2} - \frac{(4a)^7}{7 \times 64 a^3}\right] = \frac{\rho}{3}\left[\frac{2^9}{5} \cdot a^4 - \frac{2^8 \cdot a^4}{7}\right]$$

$$\boxed{I = \frac{2^8 \cdot 3\,a^4 \rho}{35}} \qquad \ldots(1)$$

Step IV : To find mass of area

$$M = \iint \rho\ dx\, dy = \rho \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy \cdot dx$$

$$M = \rho \int_{x=0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx = \rho\left[2\sqrt{a} \cdot \frac{x^{3/2}}{3/2} - \frac{x^3}{12\,a}\right]_0^{4a}$$

$$= \rho\left[2^2\, a^{1/2}\,(4a)^{3/2} - \frac{(4a)^3}{12a}\right] = \rho\left[\frac{2^5}{3} \cdot a^2 - \frac{16}{3} a^2\right]$$

$$\boxed{M = \frac{16\,a^2}{3}\,\rho} \qquad \ldots(2)$$

$\Rightarrow$ Put $2^4 \cdot a^2 \rho = 3\,M$ in Equation (1)

Step V :

$$I = \frac{2^4 \cdot 3}{35} \cdot \mathbf{a^2\, 3M}$$

$$\boxed{I = \frac{144}{35} Ma^2}$$